Group Theory Homework 3 Solutions

Karthik Boyareddygari
Professor Alexandru Suciu

Problem 1:

(1)

With $\mathbb{R}^2$ illustrated by the Euclidean plane, H is the subset having y-coordinate of 0. This is nothing more than the x-axis.

(2)

Considering $\mathbb{R}$ as a group under addition and $\mathbb{R}^2$ as the direct product $\mathbb{R} \times \mathbb{R}$, we find that $H \leq \mathbb{R}^2$ if and only if $u - v \in H$ for any $u, v \in H$. H is non-empty because every vector of the form $(x, 0) \in H$ for some $x \in \mathbb{R}$; i.e., $H = \mathbb{R} \times \{0\}$ which is non-empty. We may now take

$$u = (x_1, 0), \quad v = (x_2, 0) \quad \Longrightarrow \quad u - v = (x_1, 0) - (x_2, 0) = (x_1 - x_2, 0) \in H,$$

where the last equality follows from the definition of the group operation in a direct product. Therefore, $\boxed{H \leq \mathbb{R}^2.}$

(3)

Cosets $(x, y) + H$ take the form of vertical translates of H, where the distance and direction of the translation is dictated by y. Put another way, the set may be written as $\mathbb{R} \times \{y\}$, where we always have a full copy of $\mathbb{R}$ in the x-component because we are simply shifting an infinite line along its length.

Problem 2:

(1)

Denote by $\iota \colon H \to G$ the identity map $\mathbb{1}_G$ on G restricted to the subgroup H. Clearly, the Kernel of a restriction will always be contained within the Kernel of the unrestricted map, so ι must have trivial Kernel because it is a restriction of $\mathbb{1}_G$, which always has trivial Kernel. Therefore, ι is injective because a map is injective if and only if it has trivial Kernel.

(2)

Begin by drawing an equilateral triangle and numbering the vertices with 1, 2, and 3 (with no repetitions). The dihedral group is defined as follows

$$D_3 = \left\langle (123), (12) \mid (123)^3 = (12)^2 = (), (123)(12) = (12)(123)^2 \right\rangle.$$

It can be checked easily that these motions are rigid and that they do, indeed, form a group. Observe that each of these elements is an element of S_3 because they are written as permutations of a 3-element set. By definition, D_3 is a subgroup of S_3 because it is a subset of S_3 that also forms a group under the operation already existing in S_3.

(3)

Because D_3 is a subgroup of S_3 part (1) guarantees that there is a map $\iota\colon D_3 \to S_3$ which is injective. Furthermore, $\mathcal{O}(D_3) = 6 = \mathcal{O}(S_3)$, so injectivity and surjectivity coincide—which is a basic result for functions between equinumerous sets. Therefore, ι is a bijective map, in other words an isomorphism.

(4)

Here we define the dihedral group in much a similar fashion, just scaled up for the increase in vertices. It is important to place the condition that the numbering of vertices must be consecutive when passing between adjacent vertices (where n and 1 are considered consecutive).

$$D_n = \left\langle (12\dots n), (12) \mid (12\dots n)^n = (12)^2 = (), (12\dots n)(12) = (12)(12\dots n)^{-1} \right\rangle$$

The fact that the generators and their inverses are rigid motions (with the relations), is sufficient to say that all their products will also be rigid motions. To see this, notice that a rigid motion by definition preserves distances between all points, so composing rigid motions also preserves distances. Finally observe that each element of D_n is a permutation and is thereby an element of S_n. Therefore, $D_n \subseteq S_n$ is a subset which forms a group under the operation of S_n, which is to say $D_n \leq S_n$.

Problem 3:

(1)

Let $a, b \in G$ be arbitrary elements. We compute

$$\phi(ab) = (ab)^n = a^n b^n = \phi(a)\phi(b),$$

where the middle equality follows from the fact that G is Abelian. Therefore, ϕ is a homomorphism.

(2)

The function $\phi\colon G \to G$ need not be a homomorphism when G is not Abelian. For example, take $G = \mathrm{GL}_2(\mathbb{R})$ and $\phi\colon G \to G$ be defined by $a \mapsto a^2$ for all $a \in G$. For ϕ to be a homomorphism, it must be that $\phi(AB) = \phi(A)\phi(B)$ for any invertible matrices A and B, so finding any pair of A and B for which equality does not hold is sufficient to say that ϕ is not a homomorphism.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$$

$$\begin{aligned} \phi(AB) &= \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} \right)^2 \\ &= \begin{bmatrix} 7 & -10 \\ 17 & -24 \end{bmatrix}^2 \\ &= \begin{bmatrix} 49 & 100 \\ 289 & 576 \end{bmatrix} \end{aligned} \qquad \begin{aligned} \phi(A)\phi(B) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}^2 \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \end{aligned}$$

Clearly $\phi(AB) \neq \phi(A)\phi(B)$ for the chosen A and B, so ϕ is not a homomorphism. This shows that, in general, ϕ need not be a homomorphism when applied to a non-Abelian group.

(3)

For $G = \mathbb{Z}_{15}^{\times}$, ϕ is a homomorphism by part (1). The definition of ϕ implies that elements of $\text{Ker}(\phi)$ are order 2 elements in addition to 1 (the identity). It is easy to find that

$$\boxed{\text{Ker}(\phi) = \{1, 4, 11, 14\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2}$$

where the isomorphism is due to the fact that the only other isomorphism type for a group of order 4 is $\mathbb{Z}_4$—which is impossible for this subgroup since it has no elements of order 4 (by construction). By Lagrange's theorem and the fundamental theorem of homomorphisms, we must have that

$$\boxed{\frac{G}{\text{Ker}(\phi)} \cong \text{Im}(\phi) \cong \mathbb{Z}_2}$$

because there are only 2 cosets of $\text{Ker}(\phi)$ within G, and the only group (up to isomorphism) of order 2 is $\mathbb{Z}_2$.

It is a fact—which will be encountered later in the course—that all finite Abelian groups may be written as direct products of cyclic groups. A consequence for this problem is that G has the isomorphism type of $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Given that $\text{Ker}(\phi)$ was not cyclic and that subgroups of cyclic groups are again cyclic, we can rule out $\mathbb{Z}_8$. And if G were isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then every element (identity excluded) would be of order 2. This would imply that $\mathcal{O}(\text{Ker}(\phi)) = 8$, which we saw was not the case. Therefore, we must have that $G \cong Z_4 \times \mathbb{Z}_2$. Rewriting the image produces the statement

$$\frac{\mathbb{Z}_4 \times \mathbb{Z}_2}{\mathbb{Z}_2 \times \mathbb{Z}_2} \cong \text{Im}(\phi) \cong \mathbb{Z}_2.$$

The significance of this, which is crucial in more advanced study of algebra, is that $\text{Ker}(\phi)$ is not a *direct summand* of G in this scenario. If you're curious, you could get into some module theory by looking up *split short exact sequences* and maybe even *projective modules* (for which this would be a non-example).

Problem 4:

(1)

If G is Abelian (written additively), then

$$Z(G) = G \Longrightarrow \text{Inn}(G) \cong \frac{G}{Z(G)} \cong \{0\},$$

whereby we may conclude that every inner automorphism is simply the identity map (conjugation by the identity). Because normal subgroups are those which are preserved under all inner automorphisms, and because every subgroup is preserved under the identity map, every subgroup must be normal.

(2)

Let $\{N_\alpha\}_{\alpha \in A}$ be a collection of subgroups normal in G. In saying that a normal subgroup M is a subgroup preserved under all inner automorphisms, what is meant is that the image of M under inner automorphisms is contained within M. Define

$$N := \bigcap_{\alpha \in A} N_\alpha$$

Let $\theta\colon G \to G$ be any inner automorphism of G and $\theta|_N\colon N \to G$ be the restriction of θ to N. By hypothesis,

$$\text{Im}(\theta|_N) \leq N_\alpha \ \forall \alpha \in A \Longrightarrow \text{Im}(\theta|_N) \leq N.$$

Because the choice of θ was arbitrary, N is preserved under all inner automorphisms, thereby demonstrating that $\underline{N \text{ is normal.}}$

(3)

Take $h \in H$ to be any element, which then has to satisfy $h \in G$ because $H \leq G$. Then $hKh^{-1} = K$ because $K \trianglelefteq G$. Therefore, $\boxed{K \trianglelefteq H}$ since this holds for any element of H.

(4)

Taking advantage of the notation used in problem 6, $\langle a^2b \rangle \trianglelefteq \langle a^3, a^2b \rangle$ despite the fact that $\langle a^2b \rangle \ntrianglelefteq D_6$.

Problem 5:

S_3 is generated by the permutations (123) and (12), so any automorphism of S_3 is determined by the destinations of these 2 elements. Writing out explicitly the elements of the group

$$S_3 = \{(), (12), (13), (23), (123), (132)\},$$

we can easily see that there are 3 elements of order 2 and 2 elements of order 3. Because isomorphisms preserve the orders of elements, there are 2 possible destinations for (123) and 3 possible for (12), whereby there are at most $3 \cdot 2 = 6$ automorphisms possible. Note also that every inner automorphism is (as the name suggests) an automorphism. In particular,

$$\mathrm{Inn}(S_3) \leq \mathrm{Aut}(S_3) \Longrightarrow \mathcal{O}(\mathrm{Inn}(S_3)) \leq \mathcal{O}(\mathrm{Aut}(S_3)).$$

Now observe that no transposition commutes with both of the 3-cycles, which implies that $Z(S_3)$ is trivial. Note that the homomorphism $G \to \mathrm{Inn}(G)$ sending each element $x \in G$ to conjugation by x has Kernel $Z(G)$, so the fundamental theorem of homomorphisms asserts that $S_3 \cong \mathrm{Inn}(S_3)$.

To complete our argument, recall that $\mathcal{O}(S_3) = 3! = 6$, so

$$6 \leq \mathcal{O}(\mathrm{Aut}(S_3)) \leq 6 \Longrightarrow \mathcal{O}(\mathrm{Aut}(S_3)) = 6.$$

Because $\mathrm{Inn}(S_3) \leq \mathrm{Aut}(S_3)$, problem 2.1 guarantees the existence of an injective map $\iota\colon \mathrm{Inn}(S_3) \hookrightarrow \mathrm{Aut}(S_3)$. In fact, ι is also surjective because the orders of its domain and codomain match, so ι is an isomorphism. Therefore, we obtain

$$\boxed{\mathrm{Aut}(S_3) \cong \mathrm{Inn}(S_3) \cong S_3.}$$

Problem 6:

Similarly to problem 2.4, we will define the dihedral group on 6 vertices as

$$D_6 = \left\langle a, b \mid a^6 = b^2 = e, ab = ba^{-1} \right\rangle.$$

Our strategy for establishing the desired isomorphism will rely on finding suitable generators in $S_3 \times \mathbb{Z}_2$ which mimic the function of a and b in D_6. Because direct products with non-cyclic groups are non-cyclic, $S_3 \times \mathbb{Z}_2$ must have at least 2 generators. We claim that

$$\boxed{S_3 \times \mathbb{Z}_2 = \langle(\sigma, 1), (\tau, 0)\rangle \text{ for } \sigma = (123) \text{ and } \tau = (12).}$$

If some arbitrary $\gamma \in S_3$ can be written as some product of powers of σ and τ, which is always possible since they generate S_3, then we obtain either $(\gamma, 0)$ or $(\gamma, 1)$ from multiplying the proposed generators together in the same manner—focusing on the 1st component—used to write γ. An easy computation shows that

$$(\sigma, 1)^3 = (e, 1),$$

Upon obtaining either $(\gamma, 0)$ or $(\gamma, 1)$, we may produce the other by applying $(e, 1)$. Therefore, we have confirmed our claim.

The first part of our strategy is complete, so now we must show that the generators of $S_3 \times \mathbb{Z}_2$ act like a and b in D_6. From the associations

$$(\sigma, 1) \mapsto a \qquad \text{and} \qquad (\tau, 0) \mapsto b,$$

we check whether the relations from the definition of D_6 still hold true. In particular,

$$a^6 = (\sigma, 1)^6 = (\sigma^6, 6[1]_2) = (e, 0) = e, \qquad b^2 = (\tau, 0)^2 = (\tau^2, 2[0]_2) = (e, 0) = e,$$
$$ab = (\sigma, 1)(\tau, 0) = (\sigma\tau, 1 + 0) = (\tau\sigma^{-1}, 1) = (\tau, 0)(\sigma^{-1}, 1) = ba^{-1};$$

the last check made use of the fact that $\sigma\tau = \tau\sigma^{-1}$ within S_3 and that $-1 = 1$ in $\mathbb{Z}_2$. The fact that $S_3 \times \mathbb{Z}_2$ is generated by 2 elements subject to the same relations as in D_6 implies that there can be no more relations in $S_3 \times \mathbb{Z}_2$ without contradicting the fact that the orders of the groups (12 for both) are equal. With that said, the associations above induce a bijective homomorphism, so the groups are isomorphic.

The (non-trivial) subgroup diagram for D_6 is shown below, with normal subgroups written with an asterisk (*). It can be easily counted that there are 14 proper subgroups, 5 of which are normal.